THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2024-25
Tutorial 5 solutions
10th October 2024

- The tutorial solutions are written for reference and proofs will be sketched briefly. You should try to fill in the details as an exercise. Please send an email to echlam@math.cuhk.edu.hk if you have any further questions.

1. Let $N \triangleleft G$, $N \cap G' = \{e\}$, pick any $n \in N$, for any $g \in G$, $x = gng^{-1}n^{-1}$ is a commutator so it lies in G'. And $gng^{-1} \in N$ by normality, so $x \in N \cap G' = \{e\}$. Therefore $gn = ng$ for arbitrary $g \in G$, i.e. $n \in Z(G)$.

2. (a) Define $\phi : G/H \cap K \to G/H \times G/K$ by $\phi(aH \cap K) = (aH, aK)$, this is well-defined because if $aH \cap K = bH \cap K$, then $a^{-1}b \in H \cap K$, so $aH = bK$ and $aK = bK$. It is clearly a homomorphism. Injectivity follows from that $aH \cap K \in \ker \phi$ if and only if $aH = H$ and $aK = K$, which is equivalent to saying $a \in H \cap K \Leftrightarrow aH \cap K = H \cap K$.

   (b) Let's consider the case when G is finite first. Recall that we have

$$|HK| = \frac{|H| \cdot |K|}{|H \cap K|}.$$

   From this, we have

$$\begin{aligned} \phi \text{ is surjective} &\iff \left|\frac{G}{H \cap K}\right| = \left|\frac{G}{H}\right| \cdot \left|\frac{G}{K}\right| \\ &\iff |HK| = \frac{|H| \cdot |K|}{|H \cap K|} = |G| \\ &\iff G = HK. \end{aligned}$$

   For the case when G is infinite, we can still argue as follows. ($\Longleftarrow$) Suppose $G = HK$, given any $(aH, bK) \in G/H \times G/K$, consider $a^{-1}b \in G$, then there exists $h \in H, k \in K$ so that $a^{-1}b = hk^{-1}$, or equivalently $ah = bk$. Then we have $\phi(ahH \cap K) = (ahH, bkK) = (aH, bK)$. Therefore ϕ is surjective.

   Conversely, suppose that ϕ is surjective, then in particular for any $g \in G$, there is some $aH \cap K$ so that $\phi(aH \cap K) = (H, gK)$. In this case, $aH = H$, so $a \in H$. And $aK = gK$, so $a^{-1}g = k \in K$. Therefore $g = ak \in HK$.

   (c) We can pick $G = \mathbb{Z}$, $H = p\mathbb{Z}$ and $K = q\mathbb{Z}$. Then $H \cap K = pq\mathbb{Z}$ and the homomorphism ϕ defined in part (a) is surjective because $HK = \mathbb{Z}$, which can be seen by the fact that $\gcd(p, q) = 1$ and so there is some $a, b \in \mathbb{Z}$ so that $ap + bq = 1$ which generates $\mathbb{Z}$. This implies that $\phi : \mathbb{Z}_{pq} \to \mathbb{Z}_p \times \mathbb{Z}_q$ is an isomorphism.

3. We can write down an explicity solvable series for B_2. It suffices to note that the set A of upper triangular matrices with diagonal entries equal to 1 forms an abelian normal subgroup of B_2, with quotient isomorphic to $\mathbb{C}^\times)^2$.

Explicitly, write

$$A = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{C} \right\} \leq B_2.$$

It is clear that A is an abelian subgroup that is isomorphic to the additive group $\mathbb{C}$. It is furthermore a normal subgroup, since

$$\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & -ca^{-1}b^{-1} \\ 0 & b^{-1} \end{pmatrix}$$

and so

$$\begin{aligned} \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & -ca^{-1}b^{-1} \\ 0 & b^{-1} \end{pmatrix} &= \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \begin{pmatrix} a^{-1} & -ca^{-1}b^{-1} + b^{-1}x \\ 0 & b^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & ab^{-1}x \\ 0 & 1 \end{pmatrix} \in A. \end{aligned}$$

Next, we define $\phi : B_2 \to (\mathbb{C}^\times)^2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{C}^\times \right\}$ where $\mathbb{C}^\times$ denote the multiplicative group of complex numbers. We take $\phi(\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. It is clear that ϕ is a surjective group homomorphism, with $\ker \phi = A$. Therefore by first isomorphism theorem, we have $B_2/A \cong (\mathbb{C}^\times)^2$. Thus the series $0 \trianglelefteq A \trianglelefteq B_2$ has abelian quotient groups, so B_2 is solvable.

4. Consider the commutator subgroup $N' = [N, N]$, it is normal in G because for $g \in G$,

$$g(n_1 n_2 n_1^{-1} n_2^{-1})g^{-1} = (gn_1g^{-1})(gn_2g^{-1})(gn_1g^{-1})^{-1}(gn_2g^{-1})^{-1}$$

is again a commutator, and hence lies in N'. Here $gn_1g^{-1}, gn_2g^{-1} \in N$ by normality of N. Now by minimality of N, we have $N' = N$ or $N = \{e\}$. The former is impossible because that implies that $N^{(k)} = N$ for all higher commutator subgroup, which means that N is not solvable, contradicting the fact that G is solvable.

Remark: Here I propose a false proof that might sound convincing, try to spot the mistake in the following argument: It is possible to obtain a composition series of G by refining the sequence $0 \trianglelefteq N \trianglelefteq G$. If N was not abelian, then in the refinement, one must be able to reduce N into smaller subgroup: i.e. there exists proper subgroup M of N so that the composition series obtained looks like $0 \trianglelefteq M \trianglelefteq ... \trianglelefteq N \trianglelefteq ... \trianglelefteq G$, which contradicts with the minimality of N.

The mistake is the following: N is minimal normal subgroup of G, but in a subnormal series, M is only assumed to be normal within N, so M does not have to be a normal subgroup of G, so in fact there is no contradiction in the above.

5. No, $\mathbb{Z} \subset \mathbb{Q}$ and $\mathbb{Z}$ has no composition series. This is easily seen by the fact that every subgroup of $\mathbb{Z}$ is given by $k\mathbb{Z}$, so any subnormal series looks like

$$\mathbb{Z} \supset k_1\mathbb{Z} \supset k_2\mathbb{Z} \supset \cdots \supset k_n\mathbb{Z} \supset 0$$

But this is never a composition series as $k_n\mathbb{Z} \cong \mathbb{Z}$ is not simple.

Now $\mathbb{Q}$ is abelian so $\mathbb{Z}$ is a normal subgroup. Therefore we conclude that $\mathbb{Q}$ cannot have a composition series.

6. $D_8 \supset \mathbb{Z}_8 = \langle r \rangle$ as a normal subgroup as it has index two. Here r denotes the generator satisfying $r^8 = e$. Then we proceed by taking $\mathbb{Z}_8 \supset \langle 2 \rangle \supset \langle 4 \rangle \supset \langle e \rangle$. This is clearly a composition series as all the quotients are isomorphic to $\mathbb{Z}_2$.

   For $\mathbb{Z}_{48}$ we proceed similarly, $48 = 2^4 \cdot 3$, so we can write $\mathbb{Z}_{48} \supset \langle 2 \rangle \supset \langle 4 \rangle \supset \langle 8 \rangle \supset \langle 16 \rangle \supset \langle e \rangle$.

7. Suppose G is solvable, consider G' generated by $ghg^{-1}h^{-1}$, then $f(G')$ is generated by $f(g)f(h)f(g)^{-1}f(h)^{-1}$. If we run over all $g, h \in G$, we also run over all $f(g), f(h) \in f(G)$, so those clearly also generates $f(G)'$, and hence $f(G') = f(G)'$. Inductively, we can see $f(G^{(k)}) = f(G)^{(k)}$. Since G is solvable, $G^{(k)} = \{e\}$ for some large enough k, this implies $f(G)^{(k)}$ is trivial for that k, whence $f(G)$ is solvable.